Problem Set 8 due May 6, at 10 PM, on Gradescope

Please list all of your sources: collaborators, written materials (other than our textbook and lecture notes) and online materials (other than Gilbert Strang's videos on OCW).

Give complete solutions, providing justifications for every step of the argument. Points will be deducted for insufficient explanation or answers that come out of the blue.

Problem 1:

For any angles α and β, consider the complex numbers:

$$z = \cos\alpha + i\sin\alpha \qquad \text{and} \qquad w = \cos\beta + i\sin\beta$$

(1) Convert z and w to polar form, and compute zw in polar form. Draw z, w and zw on a picture of the complex plane, indicating their absolute value and arguments. *(10 points)*

(2) Convert zw into its Cartesian form, and use this fact to obtain formulas for:

$$\cos(\alpha+\beta) = ...$$
$$\sin(\alpha+\beta) = ...$$

in terms of $\cos\alpha$, $\sin\alpha$, $\cos\beta$, $\sin\beta$. *(10 points)*

Solution: (1) We have:

$$z = e^{i\alpha} \quad \text{and} \quad w = e^{i\beta} \quad \Rightarrow \quad zw = e^{i(\alpha+\beta)}$$

The picture must indicate the angle between the vectors z, w, zw and the positive horizontal axis.

Grading Rubric: 4 points for the correct conversion of z and w in polar, 3 points for the correct formula for zw in polar, 3 points for the picture (-1 point if the arguments, and -1 point if the absolute values of the complex numbers aren't obvious from the picture).

Solution: (2) Converting from Cartesian to polar gives us:

$$zw = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$$

On the other hand, directly multiplying z and w in Cartesian coordinates gives us:

$$zw = (\cos\alpha\cos\beta - \sin\alpha\sin\beta) + i(\cos\alpha\sin\beta + \sin\alpha\cos\beta)$$

Equating the real and imaginary parts of the formulas above gives us:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$
$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$$

Grading Rubric: 3 points for the correct Cartesian form of zw, 4 points for the correct formula form zw in terms of $\cos\alpha, \sin\alpha, \cos\beta, \sin\beta$, and 3 points for the conclusion (-1 point for minor computation errors).

Problem 2:

Fix numbers a and b. Write the symmetric matrix:

$$S = \begin{bmatrix} 0 & 0 & a & 0 \\ 0 & 0 & 0 & b \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \end{bmatrix}$$

explicitly as $Q\Lambda Q^T$, where Q is orthogonal and Λ is diagonal. Explain all of your steps (*Hint: the characteristic polynomial of a* 4×4 *matrix is a degree 4 polynomial, and therefore difficult in general to solve; however, in the case at hand, it will be easily possible to find its roots*) *(20 points)*

Solution: We compute the characteristic polynomial of the matrix as:

$$\begin{aligned} \det(S - \lambda I) &= \det \begin{bmatrix} -\lambda & 0 & a & 0 \\ 0 & -\lambda & 0 & b \\ a & 0 & -\lambda & 0 \\ 0 & b & 0 & -\lambda \end{bmatrix} \\ &= -\lambda \det \begin{bmatrix} -\lambda & 0 & b \\ 0 & -\lambda & 0 \\ b & 0 & -\lambda \end{bmatrix} + a \det \begin{bmatrix} 0 & a & 0 \\ -\lambda & 0 & b \\ b & 0 & -\lambda \end{bmatrix} \end{aligned}$$

where we have expanded along the first column. Expanding each 3×3 deteminant yields:

$$\begin{aligned} &= -\lambda \left(-\lambda \det \begin{bmatrix} -\lambda & 0 \\ 0 & -\lambda \end{bmatrix} + b \det \begin{bmatrix} 0 & -\lambda \\ b & 0 \end{bmatrix} \right) \det + a \left(-a \det \begin{bmatrix} -\lambda & b \\ b & -\lambda \end{bmatrix} \right) \\ &= \lambda^4 - \lambda^2 b^2 - a^2\lambda^2 + a^2 b^2 \\ &= (\lambda^2 - a^2)(\lambda^2 - b^2) \\ &= (\lambda - a)(\lambda + a)(\lambda - b)(\lambda + b). \end{aligned}$$

From which we see the eigenvalues are $\lambda = \pm a, \pm b$. Now we can read off the eigenvectors, e.g. by doing Gauss-Jordan elimination, which will be particularly simple for the matrix S:

$$N(S + aI) = N \begin{bmatrix} a & 0 & a & 0 \\ 0 & a & 0 & b \\ a & 0 & a & 0 \\ 0 & b & 0 & a \end{bmatrix} = \mathbb{R} \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad N(S - aI) = N \begin{bmatrix} -a & 0 & a & 0 \\ 0 & -a & 0 & b \\ a & 0 & -a & 0 \\ 0 & b & 0 & -a \end{bmatrix} = \mathbb{R} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$N(S + bI) = N \begin{bmatrix} b & 0 & a & 0 \\ 0 & b & 0 & b \\ a & 0 & b & 0 \\ 0 & b & 0 & b \end{bmatrix} = \mathbb{R} \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \qquad N(S - bI) = N \begin{bmatrix} -b & 0 & a & 0 \\ 0 & -b & 0 & b \\ a & 0 & -b & 0 \\ 0 & b & 0 & -b \end{bmatrix} = \mathbb{R} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

The change of basis matrix (from the eigenbasis to the standard basis) is therefore:

$$V = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Notice then that V has orthogonal columns, but they are not yet orthonormal (as they must be for an orthogonal matrix). To achieve orthonormality, we need to set:

$$Q = \frac{V}{\sqrt{2}} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Then the diagonalization of S is:

$$\begin{aligned} S &= VDV^{-1} = \left(\sqrt{2}Q\right) D \left(\frac{Q^{-1}}{\sqrt{2}}\right) = QDQ^T \\ &= \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -a & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & b \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} \end{aligned}$$

Grading Rubric:

- Complete and correct answer with all the steps *(20 points)*
- Correct answer, but did not provide sufficient details on how to compute either the characteristic polynomial or the eigenvectors. *(14 points)*
- Correct answer, but did not provide sufficient details on how to compute both the characteristic polynomial and eigenvectors *(8 points)*

  (in both bullets above, give the same number of points if the method was explained, but the answer for the characteristic polynomial and/or eigenvectors turned out to be significantly off)

- between -2 and -4 points for minor computational errors
- -3 points if student did not rescale the columns of Q to have length 1
- Missing or significantly incorrect answer *(0 points)*

Problem 3:

Consider the 3×3 symmetric matrix S such that:

$$\begin{bmatrix} x & y & z \end{bmatrix} S \begin{bmatrix} x \\ y \\ z \end{bmatrix} = (3x - 2y + z)^2$$

for any x, y, z.

(1) Without doing any computations on S, explain why S cannot have full rank. *(5 points)*

(2) Write S out explicitly. *(5 points)*

(3) Compute the eigenvalues and eigenvectors of S. *(10 points)*

(4) Does your answer in part (3) agree with part (1)? Explain why. Is S positive definite, positive semi-definite, or neither? *(5 points)*

Solution: (1) The quantity $(3x - 2y + z)^2$ is the energy of the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ with respect to the matrix S. Because it is non-negative for all x, y, z, then S is positive semidefinite. But because it can be 0 for x, y, z not all 0 (for example for $x = 1, y = 0, z = -3$), we conclude that S cannot be positive definite. Hence S has a zero eigenvalue, hence it cannot have full rank.

Grading Rubric:

- The explanation above, or anything similar to it (but without using computations) *(5 points)*
- No answer, wrong answer, or using computations to get the rank *(0 points)*

Solution: (2) When multiplying out the expression:

$$\begin{bmatrix} x & y & z \end{bmatrix} S \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

we see the entry S_{11} contributes the coefficient of x^2 to the expression. Likewise S_{22}, S_{33} contribute the y^2, z^2 terms. S_{12} contributes an xy term, as does S_{21}. Together, these must form the coefficient of xy in $(3x - 2y + z)^2$ and since we know that $S_{12} = S_{21}$, both S_{12} and S_{21} must be equal to the coefficient of xy. The same applies for the other off-diagonal entries. Since we have:

$$(3x - 2y + z)^2 = 9x^2 + 4y^2 + z^2 - 12xy + 6xz - 4yz \tag{1}$$

Therefore:

$$S = \begin{bmatrix} 9 & -6 & 3 \\ -6 & 4 & -2 \\ 3 & -2 & 1 \end{bmatrix}$$

Grading Rubric:

- Correct answer with the explanation above, or anything similar to it *(5 points)*
- Correct answer but without the explanation above, but with formula (1) *(4 points)*
- Correct answer but without any explanation *(3 points)*
- No answer or significantly wrong answer *(0 points)*

Solution: (3) We can calculate the characteristic polynomial (say by using cofactor expansion):

$$\det(S - \lambda I) = \det \begin{bmatrix} 9-\lambda & -6 & 3 \\ -6 & 4-\lambda & -2 \\ 3 & -2 & 1-\lambda \end{bmatrix} = \lambda^2(14-\lambda)$$

We conclude that the eigenvalues are $\lambda_1 = 14$ with multiplicity 1, and $\lambda_2 = 0$ with multiplicity 2. The eigenvectors for these eigenvalues are respectively equal to:

$$N \begin{bmatrix} 9 & -6 & 3 \\ -6 & 4 & -2 \\ 3 & -2 & 1 \end{bmatrix} = \mathbb{R} \begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} + \mathbb{R} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \qquad \text{and} \qquad N \begin{bmatrix} -5 & -6 & 3 \\ -6 & -10 & -2 \\ 3 & -2 & -13 \end{bmatrix} = \mathbb{R} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

(students should indicate how to compute the eigenvectors in detail, e.g. by running Gauss-Jordan elimination to compute nullspaces).

Grading Rubric: 4 points for the eigenvalues, and 2 points for each eigenvector (take off half the points for insufficient explanation of the algorithm, e.g. cofactor expansion or Gauss-Jordan elimination).

Solution: (4) S is positive semi-definite, since all the eigenvalues are non-negative. It is not positive definite since some of them are 0. This is in agreement with our conclusions in part 1, since positive semi-definite matrices that are not positive definite must have 0 as an eigenvalue. A square matrix fails to have full rank precisely when it has a zero eigenvalue.

Grading Rubric

- Complete explanation *(5 points)*
- Partial explanation *(3 points)*
- Missing, or very incorrect explanation *(0 points)*

Problem 4:

Represent the US flag as a 13×25 matrix A, where each entry represents a color as follows: the entry 1 represents red, the entry 0 represents white, and the entry -1 represents blue. Then write this matrix A as a sum of rank 1 matrices.

Note on vexillology: you may ignore the stars, so just assume that the top left corner is a full-blue 7×10 submatrix of A. The height of all the stripes is one row. *(15 points)*

Solution: Let's write $+$ for 1 and $-$ for -1. Then A is the following matrix:

$$\begin{bmatrix}
- & - & - & - & - & - & - & - & - & - & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
- & - & - & - & - & - & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & - & - & - & - & - & - & - & - & - & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
- & - & - & - & - & - & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & - & - & - & - & - & - & - & - & - & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
- & - & - & - & - & - & - & - & - & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
- & - & - & - & - & - & - & - & - & - & + & + & + & + & + & + & + & + & + & + & + & + & + & + & + \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+ & + \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+ & + \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
+ & +
\end{bmatrix}$$

We have:

$$A = B + C$$

where:

$$B = \frac{\boldsymbol{e}_1 + \boldsymbol{e}_3 + \boldsymbol{e}_5 + \boldsymbol{e}_7 + \boldsymbol{e}_9 + \boldsymbol{e}_{11} + \boldsymbol{e}_{13}}{\sqrt{7}} \cdot 5\sqrt{7} \cdot \frac{\boldsymbol{e}_1^T + \cdots + \boldsymbol{e}_{25}^T}{5}$$

$$C = \frac{2\boldsymbol{e}_1 + \boldsymbol{e}_2 + 2\boldsymbol{e}_3 + \boldsymbol{e}_4 + 2\boldsymbol{e}_5 + \boldsymbol{e}_6 + 2\boldsymbol{e}_7}{\sqrt{19}} \cdot (-\sqrt{190}) \cdot \frac{\boldsymbol{e}_1^T + \cdots + \boldsymbol{e}_{10}^T}{\sqrt{10}}$$

The rank 1 matrices are obtained by foiling out B and C as sums of products of the form "column times row". You don't necessarily need to renormalize the vectors above in order to have length 1.

Grading Rubric: 5 points for the correct flag (-2 points for minor errors) and 10 points for the correct decomposition as a sum of rank 1 matrices. The latter points are assigned as follows:

- Correct answer with explanation of why the matrices are rank 1 (e.g. writing them as column times row) *(10 points)*
- Correct answer without explanation of why the matrices are rank 1 *(7 points)*
- Missing, or very incorrect explanation *(0 points)*

Problem 5:

All matrices in this problem are 2×2. A lower/upper triangular matrix with 1's on the diagonal has one degree of freedom (the bottom-left/top-right entry); a diagonal matrix has two degrees of freedom (the diagonal entries). Hence the LDU factorization has $1 + 2 + 1$ degrees of freedom, which is precisely the number of degrees of freedom in choosing a 2×2 matrix.

(1) How many degrees of freedom does an orthogonal 2×2 matrix Q have? Explain. *(5 points)*

(2) What is the total number of degrees of freedom of the QR factorization? What about the total number of degrees of freedom of the SVD $U\Sigma V^T$? Explain. *(6 points)*

(3) What is the total number of degrees of freedom of $Q\Lambda Q^T$, where Q is orthogonal and Λ is diagonal? Still in the 2×2 case. *(4 points)*

(4) Why didn't you get 4 in part (3)?
Hint: it's because matrices $Q\Lambda Q^T$ are special, i.e. they are ______________ *(5 points)*

Solution: (1) There are 2 degrees of freedom in choosing a vector in the plane, but only one degree of freedom in choosing a vector of length 1 (it has to lie on a circle of radius 1 centered at the origin). Therefore, you have 1 degree of freedom in choosing the first column of an orthogonal matrix Q. But then, you have no more degrees of freedom in choosing the second column, because it has to be a length 1 vector perpendicular to the already chosen first column. Therefore, the answer is 1.

We will also accept the answer that a 2×2 orthogonal matrix is a rotation matrix, and the only degree of freedom you get to choose is an angle θ.

Grading Rubric

- Correct answer with explanation *(5 points)*
- Correct answer with insufficient explanation *(3 points)*
- Incorrect answer *(0 points)*

Solution: (2) There are 3 degrees of freedom in choosing the upper triangular matrix R with no restriction on the diagonal entries (namely the $(1,1)$, $(1,2)$ and $(2,2)$ entries). Therefore, the total number of degrees of freedom in the QR factorization is $1 + 3 = 4$, which is the same as the total number of degrees of freedom in choosing a 2×2 matrix.

Meanwhile, there are $1 + 2 + 1 = 4$ degrees of freedom in choosing the SVD $U\Sigma V^T$, since U and V must be orthogonal matrices and Σ must be diagonal.

Grading Rubric: 3 points for each of QR and SVD, each as follows:

- Correct answer with explanation *(3 points)*
- Correct answer with insufficient explanation *(2 points)*
- Incorrect answer *(0 points)*

Solution: (3) The total number of degrees of freedom is $1 + 2$: one for Q and two for Λ.

Grading Rubric:

- Correct answer with explanation *(4 points)*
- Correct answer with insufficient explanation *(2 points)*
- Incorrect answer *(0 points)*

Solution: (4) We got $1+2 = 3$ in part (c) because matrices of the form $Q\Lambda Q^T$ are not general 2×2 matrices, but just symmetric ones. You only have 3 degrees of freedom in choosing a symmetric 2×2 matrix, because the $(1, 2)$ entry must be equal to the $(2, 1)$ entry.

Grading Rubric:

- Correct answer with explanation *(5 points)*
- Correct answer with insufficient explanation *(3 points)*
- Incorrect answer *(0 points)*